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Abstract

We prove that the n^{th} pure braid group of a nonorientable surface (closed or with boundary, but different from $\mathbb{R}\mathrm{P}^2$) is residually 2-finite. Consequently, this group is residually nilpotent. The key ingredient in the closed case is the notion of p-almost direct product, which is a generalization of the notion of almost direct product. We prove therefore also some results on lower central series and augmentation ideals of p-almost direct products.

1 Introduction

Let M be a surface (orientable or not, possibly with boundary) and $F_n(M) = \{(x_1, \dots, x_n) \in M^n \,/\, x_i \neq x_j \text{ for } i \neq j\}$ its n^{th} configuration space. The fundamental group $\pi_1(F_n(M))$ is called the n^{th} *pure braid group* of M and shall be denoted by $P_n(M)$.

The mapping class group $\Gamma(M)$ of M is the group of isotopy classes of homeomorphisms $h : M \longrightarrow M$ which are identity on the boundary. Let $\mathcal{X}_n = \{z_1, \dots, z_n\}$ a set of n distinguished points in the interior of M; the pure mapping class group $\mathrm{P}\Gamma(M, \mathcal{X}_n)$ relatively to $\mathcal{X}_n$ is the group of the isotopy classes of homeomorphisms $h : M \longrightarrow M$ satisfying $h(z_i) = z_i$ for all i: since this group does not depend on the choice of the set $\mathcal{X}_n$ but only on its cardinality we can write $\mathrm{P}_n\Gamma(M)$ instead of $\mathrm{P}\Gamma(M, \mathcal{X}_n)$. Forgetting the marked points, we get a morphism $\mathrm{P}_n\Gamma(M) \longrightarrow \Gamma(M)$ whose kernel is known to be isomorphic to $P_n(M)$ when M is not a sphere, a torus, a projective plane or a Klein bottle (see [Sc, GJ]).

Now, recall that if $\mathcal{P}$ is a group-theoretic property, then a group G is said to be *residually* $\mathcal{P}$ if, for all $g \in G$, $g \neq 1$, there exists a group homomorphism $\varphi : G \longrightarrow H$ such that H is $\mathcal{P}$ and $\varphi(g) \neq 1$. We are interested in this paper to the following properties: nilpotence, being free and being a finite p-group for a prime number p (mostly $p = 2$). Recall that, if for subgroups H and K of G, $[H, K]$ is the subgroup generated by $\{[h, k] \,/\, (h, k) \in H \times K\}$ where $[h, k] = h^{-1}k^{-1}hk$, the lower central series of G, $(\Gamma_k G)_{k \geq 1}$, is defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. It is well known that G is residually nilpotent if, and only if, $\bigcap_{k=1}^{+\infty} \Gamma_k G = \{1\}$. From the lower central series of G one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \dots$ setting $D_1(G) = G$, and for $i \geq 2$, defining $D_i(G) = \{\, x \in G \,|\, \exists n \in \mathbb{N}^*, x^n \in \Gamma_i(G) \,\}$. After Garoufalidis and Levine [GLe], this filtration is called *rational lower central series* of G and a group G is residually torsion-free nilpotent if, and only if, $\bigcap_{i=1}^{\infty} D_i(G) = \{1\}$.

When M is an orientable surface of positive genus (possibly with boundary) or a disc with holes, it is proved in [BGG] and [BB] that $P_n(M)$ is residually torsion-free nilpotent for all $n \geq 1$. The fact that a

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group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable [MR] and residually p-finite [Gr]. The goal of this article is to study the nonorientable case and, more precisely, to prove the following:

Theorem 1 *The n^{th} pure braid group of a nonorientable surface different from $\mathbb{R}\mathrm{P}^2$ is residually 2-finite.*

In the case of $P_n(\mathbb{R}\mathrm{P}^2)$ we give some partial results at the end of Section 4. Since a finite 2-group is nilpotent, a residually 2-finite group is residually nilpotent. Thus, we have

Corollary 1 *The n^{th} pure braid group of a nonorientable surface different from $\mathbb{R}\mathrm{P}^2$ is residually nilpotent.*

Remark that in [Go] it was shown that the n^{th} pure braid group of a nonorientable surface is not bi-orderable and therefore it is not residually torsion-free nilpotent. Let us notice also that if pure braid groups of nonorientable surfaces with boundary are residually p for a prime $p \neq 2$ therefore pure braid groups of nonorientable closed surfaces are also residually p (Remark 3); however since the technique proposed in the nonorientable case applies only for $p = 2$, the question if pure braid groups of nonorientable surfaces are residually p for $p \neq 2$ is still open (recall that there are groups residually p for infinitely many primes p which are not residually torsion-free nilpotent, see [H]).

Remark that one can prove that finite type invariants separate classical braids using the fact that the pure braid group P_n is residually nilpotent without torsion (see [Pa]). Moreover using above residual properties it is possible to construct algebraically a universal finite type invariant over $\mathbb{Z}$ on the classical braid group B_n (see [Pa]). Similar constructions were afterwards proposed for braids on orientable surfaces (see [BF, GP]): in a further paper we will explore the relevance of Theorem 1 in the realm of finite type invariants over $\mathbb{Z}/2\mathbb{Z}$ for braids on non orientable surfaces.

From now on, $M = N_{g,b}$ is a nonorientable surface of genus g with b boundary components, simply denoted by N_g when $b = 0$. We will see N_g as a sphere S^2 with g open discs removed and g Möbius strips glued on each circle (see figure 2 where each crossed disc represents a Möbius strip). The surface $N_{g,b}$ is obtained from N_g by removing b open discs. The mapping class groups $\Gamma(N_{g,b})$ and pure mapping class group $P_n\Gamma(N_{g,b})$ will be denoted respectively $\Gamma_{g,b}$ and $\Gamma^n_{g,b}$.

The paper is organized as follows. In Section 2, we prove Theorem 1 for surfaces with boundary: following what the authors did in the orientable case (see [BGG]), we embed $P_n(N_{g,b})$ in a Torelli group. The difference here is that we must consider mod 2 Torelli groups. In Section 3 we introduce the notion of p-almost direct product, which generalizes the notion of almost direct product (see Definition 1) and we prove some results on lower central series and augmentations ideals of p-almost direct products (Theorems 4 and 5) that can be compared with similar results on almost direct products (Theorem 3.1 in [FR] and Theorem 3.1 in [Pa]).

In Section 4, the existence of a split exact sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

and results from Section 2 and 3 are used to prove Theorem 1 in the closed case (Theorem 7). The method is similar to the one developed for orientable surface in [BB]: the difference will be that the semi-direct product $P_{n-1}(N_{g,1}) \rtimes \pi_1(N_g)$ is a 2-almost-direct product (and not an almost-direct product as in the case of closed oriented surfaces).

For proving the main result of the paper, we will also need a group presentation for $P_n(N_{g,b})$ when $b \geq 1$. Although generators of this group seem to be known, we could not find a group presentation in the literature. Thus, we give one in the Appendix (Theorem A).

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2 The case of non-empty boundary

In this section, $N = N_{g,b}$ is a nonorientable surface of genus $g \geq 1$ with boundary (*ie* $b \geq 1$). In this case, one has $P_n(N) = \mathrm{Ker}(\Gamma^n_{g,b} \longrightarrow \Gamma_{g,b})$ for all $n \geq 1$.

2.1 Notations

We will follow notations from [PS]. A simple closed curve in N is an embedding $\alpha : S^1 \longrightarrow N \setminus \partial N$. Such a curve is said two-sided (resp. one-sided) if it admits a regular neighborhood homeomorphic to an annulus (resp. a Möbius strip). We shall consider the following elements in $\Gamma_{g,b}$.

- If α is a two-sided simple closed curve in N, τ_α is a Dehn twist along α.

- Let μ and α be two simple closed curves such that μ is one-sided, α is two-sided and $|\alpha \cap \mu| = 1$. A regular neighborhood K (resp. M) of $\alpha \cup \mu$ (resp. μ) is diffeomorphic to a Klein bottle with one hole (resp. a Möbius strip). Pushing M once along α, we get a diffeomorphism of K fixing the boundary: it can be extended via the identity to N. Such a diffeomorphism is called a crosscap slide, and denoted by $Y_{\mu,\alpha}$.

- Consider a one-sided simple closed curve μ containing exactly one marked point z_i. Sliding z_i once along μ, we get a diffeomorphism S_μ of N which is identity outside a regular neighborhood of μ. Such a diffeomorphism will be called puncture slide along μ.

2.2 Blowup homomorphism

In this section, we recall the construction of the Blowup homomorphism $\eta^n_{g,b} : \Gamma^n_{g,b} \longrightarrow \Gamma_{g+n,b}$ given in [Sz1], [Sz2] and [PS].

Let $U = \{z \in \mathbb{C} \,/\, |z| \leq 1\}$ and, for $i = 1, \ldots, n$, fix an embedding $e_i : U \longrightarrow N$ such that $e_i(0) = z_i$, $e_i(U) \cap e_j(U) = \emptyset$ if $i \neq j$ and $e_i(U) \cap \partial N = \emptyset$ for all i. If we remove the interior of each $e_i(U)$ (thus getting the surface $N_{g,b+n}$) and identify, for each $z \in \partial U$, $e_i(z)$ with $e_i(-z)$, we get a nonorientable surface of genus $g + n$ with b boundary components, that is to say a surface homeomorphic to $N_{g+n,b}$. Let us denote by $\gamma_i = e_i(S^1)$ the boundary of $e_i(U)$, and by μ_i its image in $N_{g+n,b}$: it is a one-sided simple closed curve.

Now, let h be an element of $\Gamma^n_{g,b}$. It can be represented by a homeomorphism $N_{g,b} \longrightarrow N_{g,b}$, still denoted h, such that

(1) $h\big(e_i(z)\big) = e_i(z)$ if h preserves local orientation at z_i;

(2) $h\big(e_i(z)\big) = e_i(\bar{z})$ if h reverses local orientation at z_i.

Such a homeomorphism h commutes with the identification leading to $N_{g+n,b}$ and thus induces an element $\eta(h) \in \Gamma_{g+n,b}$. It is proved in [Sz2] that the map $\eta^n_{g,b} = \eta : \Gamma^n_{g,b} \longrightarrow \Gamma_{g+n,b}$ which sends h to $\eta(h)$ is well defined for $n = 1$, but the proof also works for $n > 1$. This homomorphism is called *blowup* homomorphism.

Proposition 1 *The blowup homomorphism* $\eta^n_{g,b} : \Gamma^n_{g,b} \longrightarrow \Gamma_{g+n,b}$ *is injective if* $(g + n, b) \neq (2, 0)$.

Remark 1 This result is proved in [Sz1] for $(g, b) = (0, 1)$, but the proof can be adapted in our case as follows.

Proof. Suppose that $h : N_{g,b} \longrightarrow N_{g,b}$ is a homeomorphism satisfying $h(z_i) = z_i$ for all i and $\eta(h) : N_{g+n,b} \longrightarrow N_{g+n,b}$ is isotopic to identity. Then h is isotopic to a map equal to identity on $e_i(U)$ for all i (otherwise, μ_i is isotopic to μ_i^{-1} since $\eta(h)(\mu_i)$ is isotopic to μ_i) and its restriction to $N_{g,b+n}$ is an element of the kernel of the natural map $\Gamma_{g,b+n} \longrightarrow \Gamma_{g+n,b}$ induced by glueing a Möbius strip on n boundary components. However, this kernel is generated by the Dehn twists along the curves γ_i (see [St, theorem 3.6]). Now, any γ_i bounds a disc with one marked point in $N_{g,b}$: the corresponding Dehn twist is trivial in $\Gamma_{g,b}$ and therefore h is isotopic to identity. $\square$

2.3 Embedding $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$

Gluing a one-holed torus onto $b-1$ boundary components of $N_{g,b}$, we get $N_{g,b}$ as a subsurface of $N_{g+2(b-1),1}$. This inclusion induces a homomorphism $\chi_{g,b} : \Gamma_{g,b} \longrightarrow \Gamma_{g+2(b-1),1}$ which is injective (see [St]). Thus, the composed map $\lambda^n_{g,b} = \chi_{g+n,b} \circ \eta^n_{g,b} : \Gamma^n_{g,b} \longrightarrow \Gamma_{g+n+2(b-1),1}$ is also injective.

Recall that the mod p Torelli group $I_p(N_{g,1})$ is the subgroup of $\Gamma_{g,1}$ defined as the kernel of the action of $\Gamma_{g,1}$ on $H_1(N_{g,1}; \mathbb{Z}/p\mathbb{Z})$. In the following we will consider in particular the case of the mod 2 Torelli group $I_2(N_{g,1})$.

Proposition 2 *If $b \geq 1$, $\lambda^n_{g,b}\big(P_n(N_{g,b})\big)$ is a subgroup of the Torelli subgroup $I_2(N_{g+n+2(b-1),1})$.*

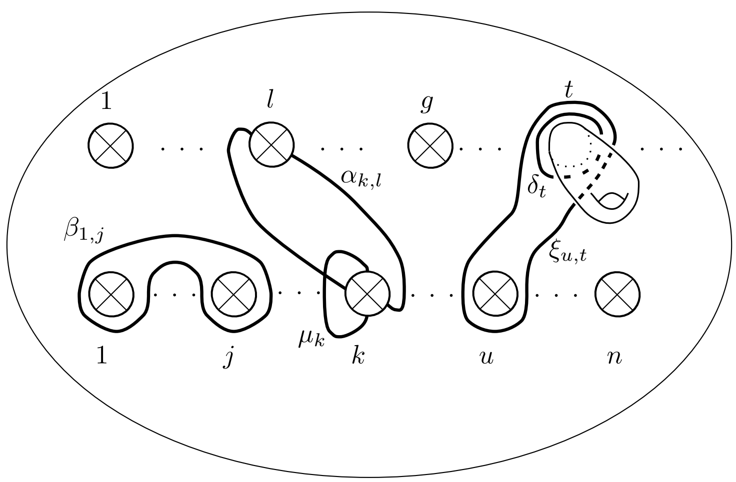

Figure 1: Image of the generators of $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$

Proof. The image of the generators (see figures 2, 6 and theorem A) $(B_{i,j})_{1\leq i<j\leq n}$, $(\rho_{k,l})_{\substack{1\leq k\leq n\\ 1\leq l\leq g}}$ and $(x_{u,t})_{\substack{1\leq u\leq n\\ 1\leq t\leq b-1}}$ of $P_n(N_{g,b})$ under $\lambda^n_{g,b}$ are respectively (see figure 1):

* Dehn twist along curves $\beta_{i,j}$, which bound a subsurface homeomorphic to $N_{2,1}$;
* crosscap slides $Y_{\mu_k,\alpha_{k,l}}$;
* the product $\tau_{\xi_{u,t}}\tau^{-1}_{\delta_t}$ of Dehn twists along the bounding curves $\xi_{u,t}$ and δ_t.

According to [Sz2], all these elements are in the mod 2 Torelli subgroup $I_2(N_{g+n+2(b-1),1})$. □

Remark 2 The embedding provided in Proposition 2 does not hold for $I_p(N_{g+n+2(b-1),1})$, when $p \neq 2$: for exemple, the cross slide $Y_{\mu_k,\alpha_{k,l}}$ is not in the mod p Torelli subgroup since it sends μ_k to μ_k^{-1}.

2.4 Conclusion of the proof

We shall use the following result, which is a straightforward consequence of a similar result for mod p Torelli groups of orientable surfaces due to L. Paris [P]:

Theorem 2 *Let $g \geq 1$. The mod p Torelli group $I_p(N_{g,1})$ is residually p-finite.*

Proof. We use Dehn-Nielsen-Baer Theorem (see for instance Theorem 5.15.3 of [CVZ]) which states that $\Gamma_{g,1}$ embeds in $Aut(\pi_1(N_{g,1}))$. Since $\pi_1(N_{g,1})$ is free we can apply Theorem 1.4 in [P] which claims that if G is a free group, its mod p Torelli group (*i.e.* the kernel of the canonical map from $Aut(G)$ to $GL(H_1(G,\mathbb{F}_p))$ is residually p-finite. Therefore $I_p(N_{g,1})$ is residually p-finite. □

Theorem 3 *Let $g \geq 1$, $b > 0$, $n \geq 1$. $P_n(N_{g,b})$ is residually 2-finite.*

Proof. The group $P_n(N_{g,b})$ is a subgroup of $I_2(N_{g+n+2(b-1),1})$ by Proposition 2 and by injectivity of the map $\lambda^n_{g,b}$. Then by Theorem 2 it follows that $P_n(N_{g,b})$ is residually 2-finite. □

3 *p*-almost direct products

3.1 On residually *p*-finite groups

Let p be a prime number and G a group. If H is a subgroup of G, we denote by H^p the subgroup generated by $\{h^p \,/\, h \in H\}$. Following [P], we define the lower $\mathbb{F}_p$-linear central filtration $(\gamma^p_n G)_{n\in\mathbb{N}^*}$ of G by $\gamma^p_1 G = G$ and, for $n \geq 1$, $\gamma^p_{n+1}G$ is the subgroup of G generated by $[G, \gamma^p_n G] \cup (\gamma^p_n G)^p$. Note that the subgroups $\gamma^p_n G$ are characteristic in G and that the quotient group $G/\gamma^p_2 G$ is nothing but the first homology group $H_1(G;\mathbb{F}_p)$. The followings are proved in [P]:

- for $m, n \geq 1$, $[\gamma^p_m G, \gamma^p_n G] \subset \gamma^p_{m+n} G$;
- a finitely generated group G is a finite p-group if, and only if, there exists some $N \geq 1$ such that $\gamma^p_N G = \{1\}$;
- a finitely generated group G is residually p-finite if, and only if, $\bigcap_{n=1}^{+\infty} \gamma^p_n G = \{1\}$;

and clearly, if $f : G \longrightarrow G'$ is a group homomorphism, then $f(\gamma^p_n G) \subset \gamma^p_n G'$ for all $n \geq 1$.

Definition 1 *Let* $1 \longrightarrow A \hookrightarrow B \underset{\sigma}{\overset{\lambda}{\rightleftarrows}} C \longrightarrow 1$ *be a split exact sequence.*

- *If the action of C induced on $H_1(A;\mathbb{Z})$ is trivial (i.e. the action is trivial on $A^{Ab} = A/[A,A]$), we say that B is an almost direct product of A and C.*
- *If the action of C induced on $H_1(A;\mathbb{F}_p)$ is trivial (i.e. the action is trivial on $A/\gamma^p_2 A$), we say that B is a p-almost direct product of A and C.*

Let us remark that, as in the case of almost direct products (Proposition 6.3 of [BGoGu]), the fact to be a p-almost direct product does not depend on the choice of the section.

PROPOSITION 3 *Let* $1 \longrightarrow A \hookrightarrow B \xrightarrow{\lambda} C \longrightarrow 1$ *be a split exact sequence of groups. Let* σ, σ' *be sections for* λ*, and suppose that the induced action of* C *on* A *via* σ *on* $H_1(A; \mathbb{F}_p)$ *is trivial. Then the same is true for the section* σ'*.*

Proof. Let $a \in A$ and $c \in C$. By hypothesis, $\sigma(c)\, a\, (\sigma(c))^{-1} \equiv a \bmod \gamma_2^p A$. Let σ' be another section for λ. Then $\lambda \circ \sigma'(c) = \lambda \circ \sigma(c)$, and so $\sigma'(c)\,(\sigma(c))^{-1} \in \mathrm{Ker}(\lambda)$. Thus there exists $a' \in A$ such that $\sigma'(c) = a'\,\sigma(c)$, and hence

$$\sigma'(c)\, a\, (\sigma'(c))^{-1} \equiv a'\,\sigma(c)\, a\, (\sigma(c))^{-1}\, a'^{-1} \equiv a' a a'^{-1} \equiv a \bmod \gamma_2^p A.$$

Thus the induced action of C on $H_1(A; \mathbb{F}_p)$ via σ' is also trivial. □

The first goal of this section is to prove the following Theorem (see Theorem 3.1 in [FR] for an analogous result for almost direct products).

THEOREM 4 *Let* $1 \longrightarrow A \hookrightarrow B \underset{\sigma}{\overset{\lambda}{\rightleftarrows}} C \longrightarrow 1$ *be a split exact sequence where* B *is a* p*-almost direct product of* A *and* C*. Then, for all* $n \geq 1$*, one has a split exact sequence*

$$1 \longrightarrow \gamma_n^p A \hookrightarrow \gamma_n^p B \underset{\sigma_n}{\overset{\lambda_n}{\rightleftarrows}} \gamma_n^p C \longrightarrow 1$$

where λ_n *and* σ_n *are restrictions of* λ *and* σ*.*

We shall need the following preliminary result.

LEMMA 1 *Under the hypotheses of Theorem 4, one has, for all* $m, n \geq 1$

$$[\gamma_m^p C', \gamma_n^p A] \subset \gamma_{m+n}^p A$$

where C' *denotes* $\sigma(C)$*.*

Proof. First, we prove by induction on n that $[C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for all $n \geq 1$. The cas $n = 1$ corresponds to the hypotheses: the action of C on $H_1(A; \mathbb{F}_p) = A/\gamma_2^p A$ is trivial if, and only if, $[C', A] \subset \gamma_2^p A$. Thus, suppose that $[C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for some $n \geq 1$ and let us prove that $[C', \gamma_{n+1}^p A] \subset \gamma_{n+2}^p A$. In view of the definition of $\gamma_{n+1}^p A$, we have to prove that $\big[C', [A, \gamma_n^p A]\big] \subset \gamma_{n+2}^p A$ and $[C', (\gamma_n^p A)^p] \subset \gamma_{n+2}^p A$. For the first case, we use a classical result (see [MKS], theorem 5.2) which says

$$\big[C', [A, \gamma_n^p A]\big] = \big[\gamma_n^p A, [C', A]\big]\big[A, [\gamma_n^p A, C']\big].$$

We have just seen that $[C', A] \subset \gamma_2^p A$ thus $\big[\gamma_n^p A, [C', A]\big] \subset [\gamma_n^p A, \gamma_2^p A] \subset \gamma_{n+2}^p A$. Then, the induction hypotheses says that $[\gamma_n^p A, C'] \subset \gamma_{n+1}^p A$ thus $\big[A, [\gamma_n^p A, C']\big] \subset [A, \gamma_{n+1}^p A] \subset \gamma_{n+2}^p A$. The second case works as follows: for $c \in C'$ and $x \in \gamma_n^p A$, one has, using the fact that $[u, vw] = [u, w][u, v]\big[[u, v], w\big]$ (see [MKS])

$$[c, x^p] = [c, x][c, x^{p-1}]\big[[c, x^{p-1}], x\big] = \cdots = [c, x]^p \big[[c, x], x\big]\big[[c, x^2], x\big] \cdots \big[[c, x^{p-1}], x\big].$$

Since $c \in C'$ and $x \in \gamma_n^p A$, one has $[c, x^i] \in [C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for all i, $1 \leq i \leq p-1$, which leads to $[c, x]^p \in (\gamma_{n+1}^p A)^p \subset \gamma_{n+2}^p A$ and $\big[[c, x^i], x\big] \in [\gamma_{n+1}^p A, A] \subset \gamma_{n+2}^p A$.

Now, we suppose that $[\gamma_m^p C', \gamma_n^p A] \subset \gamma_{m+n}^p A$ for some $m \geq 1$ and all $n \geq 1$ and prove that $[\gamma_{m+1}^p C', \gamma_n^p A] \subset \gamma_{m+n+1}^p A$. As above, there are two cases which work on the same way:

(i)
$$\begin{array}{rcl} \big[[C', \gamma_m^p C'], \gamma_n^p A\big] & = & \big[[\gamma_n^p A, C'], \gamma_m^p C'\big]\big[[\gamma_m^p C', \gamma_n^p A], C'\big] \\ & \subset & \big[\gamma_{n+1}^p A, \gamma_m^p C'\big]\big[\gamma_{m+n}^p A, C'\big] \\ & \subset & \gamma_{m+n+1}^p A. \end{array}$$

(ii) For $c \in \gamma_m^p C'$ and $x \in \gamma_n^p A$, one has

$$[c^p, x] = \big[c, [x, c^{p-1}]\big][c^{p-1}, x][c, x] = \cdots = \big[c, [x, c^{p-1}]\big] \cdots \big[c, [x, c]\big][c, x]^p$$

which is an element of $\gamma_{m+n+1}^p A$ by induction hypotheses. $\square$

Proof of Theorem 4. Restrictions of λ and σ give rise to morphisms $\lambda_n : \gamma_n^p B \longrightarrow \gamma_n^p C$ and $\sigma_n : \gamma_n^p C \longrightarrow \gamma_n^p B$ satisfying $\lambda_n \circ \sigma_n = \mathrm{Id}_{\gamma_n^p C}$, σ_n is onto and $\mathrm{Ker}(\lambda_n) = A \cap \gamma_n^p B$. Thus, we need to prove that $A \cap \gamma_n^p B = \gamma_n^p A$ for all $n \geq 1$. Clearly one has $\gamma_n^p A \subset A \cap \gamma_n^p B$. In order to prove the converse inclusion, we follow the method developed in [FR] for almost semi-direct product and define $\tau : B \longrightarrow B$ by $\tau(b) = \big(\sigma\lambda(b)\big)^{-1} b$. This map has the following properties:

(i) since $\lambda\sigma = \mathrm{Id}_C$, $\tau(B) \subset A$;

(ii) for $x \in B$, $\tau(x) = x$ if, and only if, $x \in A$;

(iii) for $(b_1, b_2) \in B^2$, $\tau(b_1 b_2) = [\sigma\lambda(b_2), \tau(b_1)^{-1}]\tau(b_1)\tau(b_2)$;

(iv) for $b \in B$, setting $a = \tau(b)$ and $c = \sigma\lambda(b)$, we get $b = ca$ with $c \in C' = \sigma(C)$ and $a \in A$, this decomposition being unique.

We claim that $\tau(\gamma_n^p B) \subset \gamma_n^p A$ for all $n \geq 1$. From this, we conclude easily the proof: if $x \in A \cap \gamma_n^p B$, then $x = \tau(x) \in \gamma_n^p A$.

One has $\tau(\gamma_1^p B) \subset \gamma_1^p A$. Suppose inductively that $\tau(\gamma_n^p B) \subset \gamma_n^p A$ for some $n \geq 1$ and let us prove that $\tau(\gamma_{n+1}^p B) \subset \gamma_{n+1}^p A$. Suppose first that x is an element of $\gamma_n^p B$. Then using *(iii)* we get:

$$\begin{array}{rcl} \tau(x^p) & = & [\sigma\lambda(x), \tau(x^{p-1})^{-1}]\tau(x^{p-1})\tau(x) \\ & \vdots & \\ & = & [\sigma\lambda(x), \tau(x^{p-1})^{-1}][\sigma\lambda(x), \tau(x^{p-2})^{-1}] \cdots [\sigma\lambda(x), \tau(x)^{-1}]\tau(x)^p. \end{array}$$

Since $\sigma\lambda(x) \in \gamma_n^p C'$ and, by induction hypotheses, $\tau(x^i) \in \gamma_n^p A$ for $1 \leq i \leq p-1$, we get $\tau(x^p) \in [\gamma_n^p C', \gamma_n^p A] \cdot (\gamma_n^p A)^p \subset \gamma_{n+1}^p A$ by lemma 1 : this prove that $\tau\big((\gamma_n^p B)^p\big) \subset \gamma_{n+1}^p A$. Next, let $b \in B$ and $x \in \gamma_n^p B$. Setting $a = \tau(b) \in A$, $y = \tau(x) \in \gamma_n^p A$ by induction hypotheses, $c = \sigma\lambda(b) \in C'$ and $z = \sigma\lambda(x) \in \gamma_n^p C'$, we get

$$\begin{array}{rcl} \tau\big([b, x]\big) & = & \Big(\sigma\lambda\big([b, x]\big)\Big)^{-1}[b, x] \\ & = & \big[\sigma\lambda(b), \sigma\lambda(x)\big]^{-1}[b, x] \\ & = & [c, z]^{-1}[ca, zy] = [z, c]a^{-1}c^{-1}y^{-1}z^{-1}cazy \\ & = & [z, c]\big(a^{-1}c^{-1}y^{-1}cya\big)\big(a^{-1}y^{-1}c^{-1}z^{-1}czya\big)\big(a^{-1}y^{-1}z^{-1}azy\big) \\ & = & [z, c]\big(a^{-1}[c, y]a\big)\big(a^{-1}y^{-1}[c, z]ya\big)\big(a^{-1}y^{-1}ay\big)\big(y^{-1}a^{-1}z^{-1}azy\big) \\ & = & [z, c]\big(a^{-1}[c, y]a\big)\big(a^{-1}y^{-1}[c, z]ya\big)[a, y]\big(y^{-1}[a, z]y\big) \\ & = & \Big[[c, z], \big(a^{-1}[y, c]a\big)\Big]\big(a^{-1}[c, y]a\big)[z, c]\big(a^{-1}y^{-1}[c, z]ya\big)[a, y]\big(y^{-1}[a, z]y\big) \end{array}$$

$$= \quad \Big[[c,z],\big(a^{-1}[y,c]a\big)\Big]\big(a^{-1}[c,y]a\big)\Big[[c,z],ya\Big][a,y]\big(y^{-1}[a,z]y\big).$$

Now, $[c,z]\in[C',\gamma_n^p C']\subset\gamma_{n+1}^p C'$, $[y,c]\in[\gamma_n^p A, C']\subset\gamma_{n+1}^p A$ (lemma 1) thus $\Big[[c,z],\big(a^{-1}[y,c]a\big)\Big]\in\gamma_{n+1}^p A$. Then, $\Big[[c,z],ya\Big]\in[\gamma_{n+1}^p C', A]\subset\gamma_{n+1}^p A$, $[a,y]\in[A,\gamma_n^p A]\subset\gamma_{n+1}^p A$ and $[a,z]\in[A,\gamma_n^p C']\subset\gamma_{n+1}^p A$. Thus, $\tau([b,x])\in\gamma_{n+1}^p A$ and $\tau([B,\gamma_n^p B])\subset\gamma_{n+1}^p A$. □

Corollary 2 *Let* $1\longrightarrow A\hookrightarrow B \underset{\sigma}{\overset{\lambda}{\rightleftarrows}} C\longrightarrow 1$ *be a split exact sequence such that B is a p-almost direct product of A and C. If A and C are residually p-finite, then B is residually p-finite.*

3.2 Augmentation ideals

Given a group G and $\mathbb{K}=\mathbb{Z}$ or $\mathbb{F}_2$ we will denote by $\mathbb{K}[G]$ the group ring of G over $\mathbb{K}$ and by $\overline{\mathbb{K}[G]}$ the augmentation ideal of G. The group ring $\mathbb{K}[G]$ is filtered by the powers $\overline{\mathbb{K}[G]}^j$ of $\overline{\mathbb{K}[G]}$ and we can define the associated graded algebra $gr(\mathbb{K}[G])=\oplus\overline{\mathbb{K}[G]}^j/\overline{\mathbb{K}[G]}^{j+1}$.

The following theorem provides a decomposition formula for the augmentation ideal of a $2-$almost direct product (see Theorem 3.1 in [Pa] for an analogous in the case of almost direct products).
Let $A\rtimes C$ be a semi-direct product between two groups A and C. It is a classical result that the map $a\otimes c\mapsto ac$ induces a $\mathbb{K}$-isomorphism from $\mathbb{K}[A]\otimes\mathbb{K}[C]$ to $\mathbb{K}[A\rtimes C]$. Identifying these two $\mathbb{K}$-modules, we have the following:

Theorem 5 *If $A\rtimes C$ is a $2-$almost direct product then :*

$$\overline{\mathbb{F}_2[A\rtimes C]}^k=\sum_{i+h=k}\overline{\mathbb{F}_2[A]}^i\otimes\overline{\mathbb{F}_2[C]}^h \qquad \textit{for all } k.$$

Proof. We sketch the proof which is almost verbatim the same as the proof of Theorem 3.1 in [Pa]. Let $R_k=\sum_{i+h=k}\overline{\mathbb{F}_2[A]}^i\otimes\overline{\mathbb{F}_2[C]}^h$; R_k is a descending filtration on $\mathbb{F}_2[A]\otimes\mathbb{F}_2[C]$, and with the above identification, we get that $R_k\subset\overline{\mathbb{F}_2[A\rtimes C]}^k$. To verify the other inclusion we have to check that $\prod_{j=1}^k(a_jc_j-1)\in R_k$ for every $a_1,\dots,a_k$ in A and $c_1,\dots,c_k$ in C. Actually it is enough to verify that $e=\prod_{j=1}^k(e_j-1)\in R_k$ either $e_j\in A$ or $e_j\in C$ (see Theorem 3.1 in [Pa] for a proof of this fact): we call e a *special* element. We associate to a special element e an element in $\{0,1\}^k$: let $type(e)=(\delta(e_1),\dots,\delta(e_k))$ where $\delta(e_j)=0$ if $e_j\in A$ and $\delta(e_j)=1$ if $e_j\in C$. We will say that the special element e is *standard* if

$$\text{type}(e)=(\overbrace{0,\dots,0}^{i},\overbrace{1,\dots,1}^{h})$$

In this case $e\in\overline{\mathbb{F}_2[A]}^i\otimes\overline{\mathbb{F}_2[C]}^h\subset R_k$ and we are done. We claim that we can reduce all special elements to linear combinations of standard elements. If e is not standard, then it must be of the form

$$e=\prod_{i=1}^r(a_i-1)\prod_{j=1}^s(c_i-1)(c-1)(a-1)\prod_{l=1}^t(e_i-1)$$

where $a_1, \ldots, a_r, a \in A$, $c_1, \ldots, c_s, c \in A$, $\tilde{e} = \prod_{l=1}^{t}(e_i - 1)$ is special and $r + s + t + 2 = k$. Therefore

$$\text{type}(e) = (\overbrace{0, \ldots, 0}^{r}, \overbrace{1, \ldots, 1}^{s}, 1, 0, \delta(e_1), \ldots, \delta(e_t))\,.$$

Now we can use the assumption that $A \rtimes C$ is a $2-$almost direct product to claim that one has commutation relations in $\mathbb{Z}[A \rtimes C]$ expressing the difference $(c-1)(a-1) - (a-1)(c-1)$ as a linear combination of terms of the form

$$(a' - 1)(a'' - 1)c \qquad \text{with } a', a'' \in A$$

for any $a \in A$ and $c \in C$. In fact,

$$(c-1)(a-1) - (a-1)(c-1) = ca - ac = (cac^{-1}a^{-1} - 1)ac = (f-1)ac$$

where $f = [c^{-1}, a^{-1}] \in [C, A] \subset \gamma_2^2(A)$ by lemma 1. We can decompose f as $f = h_1 k_1 \cdots h_m k_m$ where, for $j = 1, \ldots, m$, h_j belongs to $[A, A]$ and $k_j = (k'_j)^2$ for some $k'_j \in A$. One knows (see for instance [Ch] p. 194) that for $j = 1, \ldots, m$ $(h_j - 1)$ is a linear combination of terms of the form

$$(h'_j - 1)(h''_j - 1)\alpha_j \qquad \text{with } h'_j, h''_j, \alpha_j \in A.$$

On the other hand for $j = 1, \ldots, m$ we have also that

$$(k_j - 1) = (k'_j - 1)(k'_j - 1) \qquad \text{with } k'_j \in A \quad \text{since the coefficients are } \mathbb{F}_2.$$

Then, recalling that $(hk - 1) = (h-1)k + (k-1)$ for any $h, k \in A$, we can conclude that $f - 1$ can be rewritten as a linear combination of terms of the form

$$(f' - 1)(f'' - 1)\alpha \qquad \text{with } f', f'', \alpha \in A$$

and that $(c-1)(a-1) - (a-1)(c-1)$ is a linear combination of terms of the form

$$(f' - 1)(f'' - 1)\alpha c \qquad \text{with } f', f'', \alpha \in A.$$

Rewriting $(f'' - 1)\alpha$ as $(f''\alpha - 1) - (\alpha - 1)$ we obtain that the difference $(c-1)(a-1) - (a-1)(c-1)$ can be seen as a linear combination of terms of the form

$$(a' - 1)(a'' - 1)c \qquad \text{with } a', a'' \in A.$$

Therefore e can be rewritten as a sum whose first term is the special element

$$e' = \prod_{i=1}^{r}(a_i - 1)\prod_{j=1}^{s}(c_i - 1)(a-1)(c-1)\prod_{l=1}^{t}(e_i - 1)$$

and whose second term is a linear combination of elements of the form $e''c$ where

$$e'' = \prod_{i=1}^{r}(a_i - 1)\prod_{j=1}^{s}(c_i - 1)(a' - 1)(a'' - 1)\prod_{l=1}^{t}(ce_ic^{-1} - 1)c.$$

Using the lexicographic order from the left, one has type(e) >type(e') and type(e) >type(e'').

By induction on the lexicographic order we infer that e' and e'' belong to R_k: since $R_k \cdot c \subset R_k$ for any $c \in C$ it follows that e belongs to R_k and we are done. □

4 The closed case

4.1 A presentation of $P_n(N_g)$ and induced identities

We recall a group presentation of $P_n(N_g)$ given in [GG3]: the geometric interpretation of generators is provided in Figure 2.

THEOREM 6 ([GG3]) *For $g \geq 2$ and $n \geq 1$, $P_n(N_g)$ has the following presentation:*

generators: $(B_{i,j})_{1\leq i<j\leq n}$ *and* $(\rho_{k,l})_{\substack{1\leq k\leq n\\ 1\leq l\leq g}}$.

relations : (a) *for all* $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j & (a_1)\\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j} & \text{if } r<i=s<j & (a_2)\\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i=r<s<j & (a_3)\\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j} & \text{if } r<i<s<j & (a_4)\end{cases}$$

(b) *for all* $1 \leq i < j \leq n$ *and* $1 \leq k, l \leq g$,

$$\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k<l & (b_1)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2} & \text{if } k=l & (b_2)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k} & \text{if } k>l & (b_3)\end{cases}$$

(c) *for all* $1 \leq i \leq n$, $\rho_{i,1}^2 \cdots \rho_{i,g}^2 = T_i$ *where* $T_i = B_{1,i}\cdots B_{i-1,i}B_{i,i+1}\cdots B_{i,n}$ (c)

(d) *for all* $1 \leq i < j \leq n$, $1 \leq k \leq n$, $k \neq j$ *and* $1 \leq l \leq g$,

$$\rho_{k,l}B_{i,j}\rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k<i \text{ or } j<k & (d_1)\\ \rho_{j,l}^{-1}B_{i,j}^{-1}\rho_{j,l} & \text{if } k=i & (d_2)\\ \rho_{j,l}^{-1}B_{k,j}^{-1}\rho_{j,l}B_{k,j}^{-1}B_{i,j}B_{k,j}\rho_{j,l}^{-1}B_{k,j}\rho_{j,l} & \text{if } i<k<j & (d_3)\end{cases}$$

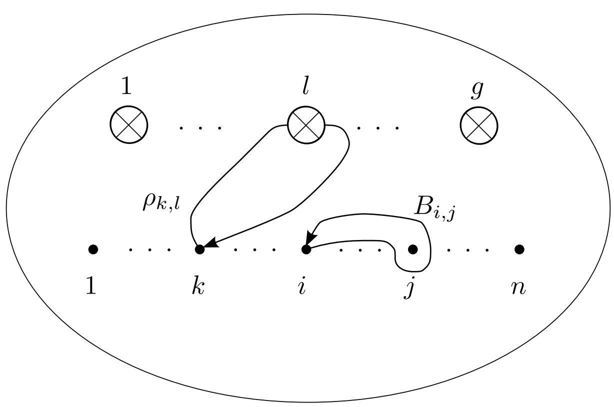

Figure 2: Generators of $P_n(N_g)$

For $1 \leq k \leq g$, let us consider the element a_k in $P_n(N_g)$ given by $a_k = \rho_{k,g-1}\rho_{k,g}$ and set $U = a_n \cdots a_2$.

LEMMA 2 *The following relations holds in $P_n(N_g)$:*

(1) $[\rho_{i,k}, \rho_{j,k}^{-1}] = B_{i,j}^{-1}$ *for* $1 \leq i < j \leq n$ *and* $1 \leq k \leq g$; (e)

(2) U *commutes with* $\rho_{1,l}$ *for* $1 \leq l \leq g-2$; (f$_1$)

(3) $[\rho_{1,g-1}, U^{-1}] = T_1^{-1}$; (f$_2$)

(4) $a_k a_j a_k^{-1}$ *commutes with* $B_{i,k}$ *for* $1 \leq i < j < k \leq n$; (g)

(5) $a_n a_{n-1} \cdots a_1$ *commutes with* $B_{j,k}$ *for* $1 \leq j < k \leq n$; (h)

(6) U *commutes with* $B_{i,j}$ *for* $2 \leq i < j \leq n$; (i)

(7) $a_n a_{n-1} \cdots a_1$ *commutes with* T_1; (j)

(8) T_1 *commutes with* $B_{j,k}$ *for* $2 \leq j < k \leq n$; (k)

Proof. Some of these identities can easily be verified drawing corresponding braids. This is the case for example for the first, the fourth and the eighth ones (see figure 3, 4 and 5). Let us give an algebraic proof for the others.

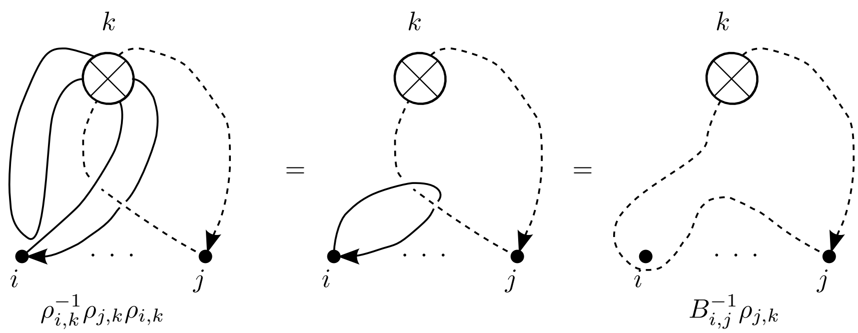

Figure 3: identity (e)

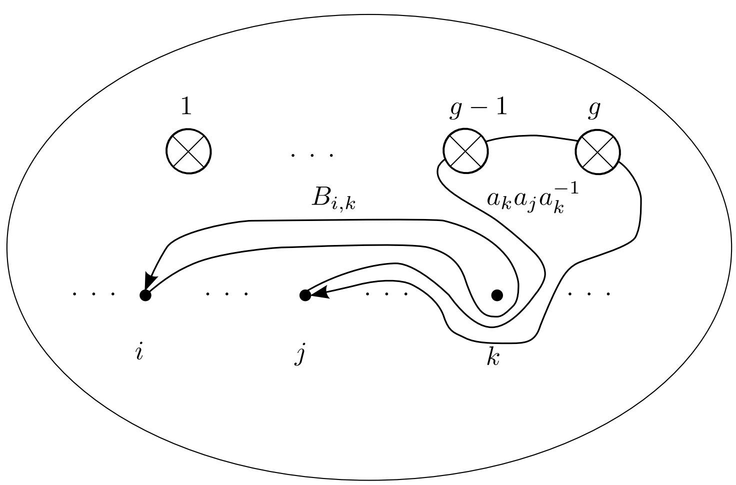

Figure 4: identity (g)

(2) This is a direct consequence of the definitions of the a_i's and U, and relations (b$_1$).

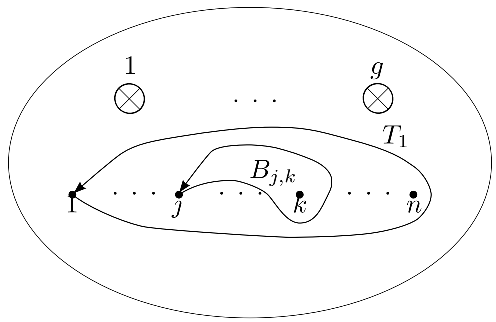

Figure 5: identity (k)

(3) By relation (b$_1$), $\rho_{1,g-1}$ commutes with $\rho_{j,g}$ for $2 \le j \le n$. Thus, using relation (e), we get:

$$\begin{aligned}
\rho_{1,g-1}^{-1} U \rho_{1,g-1} &= \rho_{1,g-1}^{-1} a_n \cdots a_2 \rho_{1,g-1} \\
&= \rho_{1,g-1}^{-1} (\rho_{n,g-1}\rho_{n,g}) \cdots (\rho_{2,g-1}\rho_{2,g}) \rho_{1,g-1} \\
&= (B_{1,n}^{-1} \rho_{n,g-1}\rho_{n,g}) \cdots (B_{1,2}^{-1} \rho_{2,g-1}\rho_{2,g}) \\
&= B_{1,n}^{-1} B_{1,n-1}^{-1} \cdots B_{1,2}^{-1} (\rho_{n,g-1}\rho_{n,g}) \cdots (\rho_{2,g-1}\rho_{2,g}) \quad \text{by } (\mathrm{d}_1) \\
&= T_1^{-1} U.
\end{aligned}$$

(5) Let j and k be integers such that $1 \le j < k \le n$. By (d$_1$), $a_1, \dots, a_{j-1}$ commute with $B_{j,k}$. Then, one has

$$\begin{aligned}
a_j B_{j,k} a_j^{-1} &= \rho_{j,g-1}\rho_{j,g} B_{j,k} \rho_{j,g}^{-1} \rho_{j,g-1}^{-1} \\
&= \rho_{j,g-1} \rho_{k,g}^{-1} B_{j,k}^{-1} \rho_{k,g} \rho_{j,g-1}^{-1} \quad \text{by } (\mathrm{d}_2) \\
&= \rho_{k,g}^{-1} \rho_{j,g-1} B_{j,k}^{-1} \rho_{j,g-1}^{-1} \rho_{k,g} \quad \text{by } (\mathrm{b}_1) \\
&= \rho_{k,g}^{-1} \rho_{k,g-1}^{-1} B_{j,k} \rho_{k,g-1} \rho_{k,g} \quad \text{by } (\mathrm{d}_2) \\
&= a_k^{-1} B_{j,k} a_k
\end{aligned}$$

and we get

$$\begin{aligned}
a_n \cdots a_1 B_{j,k} a_1^{-1} \cdots a_n^{-1} &= a_n \cdots a_{k+1} a_k a_{k-1} \cdots a_{j+1} a_k^{-1} B_{j,k} a_k a_{j+1}^{-1} \cdots a_{k-1}^{-1} a_k^{-1} a_{k+1}^{-1} \cdots a_n^{-1} \\
&= a_n \cdots a_{k+1} B_{j,k} a_{k+1}^{-1} \cdots a_n^{-1} \quad \text{by (g)} \\
&= B_{j,k} \quad \text{by } (\mathrm{d}_1).
\end{aligned}$$

(6) By (d_1), $a_1 = \rho_{1,g-1}\rho_{1,g}$ commutes with $B_{i,j}$ for $2 \le i < j \le n$. Thus, relation (i) is a direct consequence of (h).

(7) A direct consequence of (h) since $T_1 = B_{1,2} \cdots B_{1,n}$.

□

4.2 The pure braid group $P_n(N_g)$ is residually 2-finite

Following [GG1], one has, for $g \ge 2$, a split exact sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \xrightarrow{\mu} P_n(N_g) \xrightarrow{\lambda} P_1(N_g) = \pi_1(N_g) \longrightarrow 1 \tag{1}$$

where λ is induced by the map which forgets all strands except the first one, and μ is defined by capping the boundary component by a disc with one marked point (the first strand in $P_n(N_g)$). According to the definition of μ and to Theorem A, $\mathrm{Im}(\mu)$ is generated by $\{\rho_{i,k},\ 2 \le i \le n,\ 1 \le k \le g\} \cup \{B_{i,j},\ 2 \le i < j \le n\}$.

The section given in [GG1] is geometric, *i.e.* it is induced by a crossed section at the level of fibrations. In order to study the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$, we need an algebraic one. Recall that $\pi_1(N_g)$ has a group presentation with generators $p_1, \dots, p_g$ and the single relation $p_1^2 \cdots p_g^2 = 1$. We define the set map $\sigma : \pi_1(N_g) \longrightarrow P_n(N_g)$ by setting

$$\sigma(p_i) = \begin{cases} \rho_{1,i} & \text{for } 1 \le i \le g-3, \\ \rho_{1,g-2}U^{-1} & \text{for } i = g-2, \\ U\rho_{1,g-1} & \text{for } i = g-1, \\ \rho_{1,g}T_1^{-1} & \text{for } i = g. \end{cases}$$

Proposition 4 *The map σ is a well defined homomorphism satisfying $\lambda \circ \sigma = \mathrm{Id}_{\pi_1(N_g)}$.*

Proof. Since $\lambda(\rho_{1,i}) = p_i$ for all $1 \le i \le g$ and $\lambda(U) = \lambda(T_1) = 1$, one has clearly $\lambda\sigma = \mathrm{Id}_{\pi_1(N_g)}$ if σ is a group homomorpism. Thus, we have just to prove that $\sigma(p_1)^2 \cdots \sigma(p_g)^2 = 1$:

$$\begin{aligned} \sigma(p_1)^2 \cdots \sigma(p_g)^2 &= (\rho_{1,1}^2 \cdots \rho_{1,g-3}^2)(\rho_{1,g-2}U^{-1})^2(U\rho_{1,g-1})^2(\rho_{1,g}T_1^{-1})^2 \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2} \underbrace{U^{-1}\rho_{1,g-2}}\, \rho_{1,g-1} U a_1 T_1^{-1} \rho_{1,g} T_1^{-1} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \underbrace{U^{-1}\rho_{1,g-1}}\, U a_1 T_1^{-1} \rho_{1,g} T_1^{-1} \quad \text{by } (\mathrm{f}_1) \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1} U^{-1} T_1 \underbrace{U a_1 T_1^{-1}}\, \rho_{1,g} T_1^{-1} \quad \text{by } (\mathrm{f}_2) \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1} U^{-1} T_1 T_1^{-1} U a_1 \rho_{1,g} T_1^{-1} \quad \text{by (j)} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1} a_1 \rho_{1,g} T_1^{-1} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1}^2 \rho_{1,g}^2 T_1^{-1} \\ &= 1 \quad \text{by (c).} \end{aligned}$$

□

So, the exact sequence (1) splits. In order to apply Theorem 4, we have to prove that the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$ is trivial on $H_1(P_{n-1}(N_{g,1}); \mathbb{F}_2)$. This is the claim of the following proposition.

Proposition 5 *For all $x \in \mathrm{Im}(\sigma)$ and $a \in \mathrm{Im}(\mu)$, one has $[x^{-1}, a^{-1}] = xax^{-1}a^{-1} \in \gamma_2^2(\mathrm{Im}(\mu))$.*

Proof. It is enough to prove the result for $a \in \{B_{j,k},\ 2 \le j < k \le n\} \cup \{\rho_{j,l},\ 2 \le j \le n \text{ and } 1 \le l \le g\}$ and $x \in \{\sigma(p_1), \dots, \sigma(p_g)\}$, respectively sets of generators of $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\sigma)$. Suppose first that $2 \le j < k \le n$. One has:

- $[\sigma(p_i)^{-1}, B_{j,k}^{-1}] = [\rho_{1,i}^{-1}, B_{j,k}^{-1}] = 1$ for $1 \le i \le g-3$ by (d_1);
- $[\sigma(p_{g-2})^{-1}, B_{j,k}^{-1}] = [U\rho_{1,g-2}^{-1}, B_{j,k}^{-1}] = 1$ by (d_1) and (i);
- $[\sigma(p_{g-1})^{-1}, B_{j,k}^{-1}] = [\rho_{1,g-1}^{-1}U^{-1}, B_{j,k}^{-1}] = 1$ by (d_1) and (i);
- $[\sigma(p_g)^{-1}, B_{j,k}^{-1}] = [T_1\rho_{1,g}^{-1}, B_{j,k}^{-1}] = 1$ by (d_1) and (k).

Now, let j and l be integers such that $2 \le j \le n$ and $1 \le l \le g$ and let us first prove that $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ for all i, $1 \le i \le n$:

- this is clear for $i < l$ by (b_1);
- for $i = l$, the relation (b_2) gives $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] = \rho_{j,l}^{-1} B_{1,j}^{-1} \rho_{j,l}$. But

  $$B_{1,j}^{-1} = B_{2,j} \cdots B_{j-1,j} B_{j,j+1} \cdots B_{j,n} \rho_{j,g}^{-2} \cdots \rho_{j,1}^{-2} \quad \text{(relation (c))}$$

  is an element of $\gamma_2^2(\mathrm{Im}(\mu))$ by (e), thus we get $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.
- If $l < i$ then $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] = [B_{1,j} \rho_{j,i}^{-1} B_{1,j} \rho_{j,i}, \rho_{j,l}^{-1}]$ by (b_3) so $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2(\mathrm{Im}(\mu))$ since $\rho_{j,l}$, $\rho_{j,i}$ and $B_{1,j}$ are elements of $\mathrm{Im}(\mu)$.

From this, we deduce the following facts.

(1) $[\sigma(p_i)^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$ for $i \leq g-3$ since $\sigma(p_i) = \rho_{1,i}$.

(2) $[\sigma(p_{g-2})^{-1}, \rho_{j,l}^{-1}] = [U \rho_{1,g-2}^{-1}, \rho_{j,l}^{-1}] = \rho_{1,g-2}[U, \rho_{j,k}^{-1}] \rho_{1,g-2}^{-1} [\rho_{1,g-2}^{-1}, \rho_{j,l}^{-1}]$. But U and $\rho_{j,l}^{-1}$ are elements of $\mathrm{Im}(\mu)$ thus $[U, \rho_{j,l}^{-1}] \in \Gamma_2(\mathrm{Im}(\mu)) \subset \gamma_2^2(\mathrm{Im}(\mu))$. Consequently, $\rho_{1,g-2}[U, \rho_{j,k}^{-1}] \rho_{1,g-2}^{-1} \in \gamma_2^2(\mathrm{Im}(\mu))$ since $\gamma_2^2(\mathrm{Im}(\mu))$ is a caracteristic subgroup of $\mathrm{Im}(\mu)$ and $\mathrm{Im}(\mu)$ is normal in $P_n(N_g)$. Thus, we get $[\sigma(p_{g-2})^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$. In the same way, one has

$$[\sigma(p_{g-1})^{-1}, \rho_{j,l}^{-1}] = [\rho_{1,g-1}^{-1} U^{-1}, \rho_{j,l}^{-1}] = U[\rho_{1,g-1}^{-1}, \rho_{j,l}^{-1}] U^{-1} [U, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu)).$$

(4) At last,

$$[\sigma(p_g)^{-1}, \rho_{j,l}^{-1}] = [T_1 \rho_{1,g}^{-1}, \rho_{j,l}^{-1}] = \rho_{1,g}[T_1, \rho_{j,l}^{-1}] \rho_{1,g}^{-1} [\rho_{1,g}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$$

since $T_1, \rho_{j,l} \in \mathrm{Im}(\mu)$ and $[\rho_{1,g}, \rho_{j,l}^{-1}] \in \gamma_2^2(\mathrm{Im}(\mu))$.

□

We are now ready to prove the main result of this section

Theorem 7 *For all $g \geq 2$ and $n \geq 1$, the pure braid group $P_n(N_g)$ is residually 2-finite.*

Proof. Proposition 4 says that the sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

splits. Now $P_{n-1}(N_{g,1})$ is residually 2-finite (Theorem 3). It is proved in [B1] and [B2] that $\pi_1(N_g)$ is residually free for $g \geq 4$, so it is residually 2-finite. This result is proved in [LM] (lemma 8.9) for $g = 3$. When $g = 2$, $\pi_1(N_2)$ has presentation $\langle a, b \,|\, aba^{-1} = b^{-1} \rangle$ so is a 2-almost direct product of $\mathbb{Z}$ by $\mathbb{Z}$. Since $\mathbb{Z}$ is residually 2-finite, $\pi_1(N_2)$ is residually 2-finite by corollary 2. So, using Proposition 5 and Corollary 2, we can conclude that $P_n(N_g)$ is residually 2-finite. □

Remark 3 It follows from the proof of Theorem 7 that, when $g > 2$, if $P_n(N_{g,1})$ is residually p-finite for some $p \neq 2$ then the pure braid group $P_n(N_g)$ is also residually p-finite.

4.3 The case $P_n(\mathbb{R}\mathrm{P}^2)$

The main reason to exclude $N_1 = \mathbb{R}\mathrm{P}^2$ in Theorem 7 is that the exact sequence (1) doesn't exist in this case, but forgetting at most $n-2$ strands we get the following exact sequence ($1 \le m \le n-2$; see [VB])

$$1 \longrightarrow P_m(N_{1,n-m}) \longrightarrow P_n(\mathbb{R}\mathrm{P}^2) \longrightarrow P_{n-m}(N_g) \longrightarrow 1 \ .$$

This sequence splits if, and only if $n = 3$ and $m = 1$ (see [GG2]). Thus, what we know is the following:

- $P_1(\mathbb{R}\mathrm{P}^2) = \pi_1(\mathbb{R}\mathrm{P}^2) = \mathbb{Z}/2\mathbb{Z}$: $P_1(\mathbb{R}\mathrm{P}^2)$ is a 2-group.
- $P_2(\mathbb{R}\mathrm{P}^2) = Q_8$, the quaternion group (see [VB]): $P_2(\mathbb{R}\mathrm{P}^2)$ is a 2-group.
- One has the exact sequence

$$1 \longrightarrow P_1(N_{1,2}) \longrightarrow P_3(\mathbb{R}\mathrm{P}^2) \longrightarrow P_2(\mathbb{R}\mathrm{P}^2) \longrightarrow 1$$

  where $P_1(N_{1,2}) = \pi_1(N_{1,2})$ is a free group of rank 2, thus is residually 2-finite. Since $P_2(\mathbb{R}\mathrm{P}^2)$ is 2-finite, we can conclude that $P_3(\mathbb{R}\mathrm{P}^2)$ is residually 2-finite using lemma 1.5 of [Gr].

Appendix: a group presentation for $P_n(N_{g,b})$

Here we apply classical method (see [B, GG3]) to give a presentation of the n^{th} pure braid group of a nonorientable surface with boundary. Since $b \ge 1$, we'll see $N_{g,b}$ as a disc D^2 with $g+b-1$ open discs removed and g Möbius strips glued on g boundary components so obtained (see figure 6).

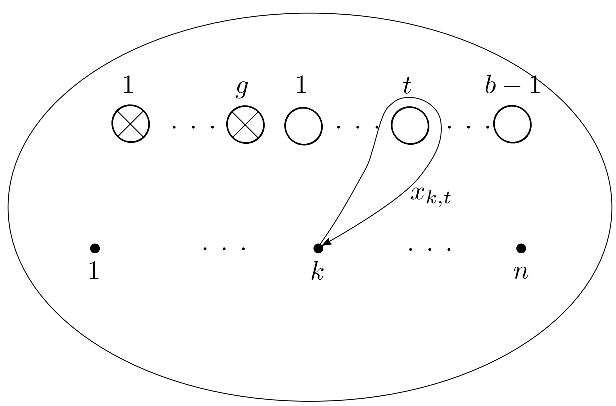

Figure 6: Generators $x_{k,t}$ for $P_n(N_{g,b})$, $b \ge 1$

Theorem A *For $g \ge 1$, $b \ge 1$ and $n \ge 1$, $P_n(N_{g,b})$ has the following presentation:*

<u>**generators:**</u> $(B_{i,j})_{1 \le i < j \le n}$, $(\rho_{k,l})_{\substack{1 \le k \le n \\ 1 \le l \le g}}$ *and* $(x_{u,t})_{\substack{1 \le u \le n \\ 1 \le t \le b-1}}$.

relations : (a) *for all* $1 \le i < j \le n$ and $1 \le r < s \le n$,

$$B_{r,s}B_{i,j}B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i<r<s<j \text{ or } r<s<i<j \quad (a_1)\\ B_{i,j}^{-1}B_{r,j}^{-1}B_{i,j}B_{r,j}B_{i,j} & \text{if } r<i=s<j \quad (a_2)\\ B_{s,j}^{-1}B_{i,j}B_{s,j} & \text{if } i=r<s<j \quad (a_3)\\ B_{s,j}^{-1}B_{r,j}^{-1}B_{s,j}B_{r,j}B_{i,j}B_{r,j}^{-1}B_{s,j}^{-1}B_{r,j}B_{s,j} & \text{if } r<i<s<j \quad (a_4)\end{cases}$$

(b) *for all* $1 \le i < j \le n$ *and* $1 \le k,l \le g$,

$$\rho_{i,k}\rho_{j,l}\rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k<l \quad (b_1)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}^{2} & \text{if } k=l \quad (b_2)\\ \rho_{j,k}^{-1}B_{i,j}^{-1}\rho_{j,k}B_{i,j}^{-1}\rho_{j,l}B_{i,j}\rho_{j,k}^{-1}B_{i,j}\rho_{j,k} & \text{if } k>l \quad (b_3)\end{cases}$$

(d) *for all* $1 \le i < j \le n,\ 1 \le k \le n,\ k \neq j$ *and* $1 \le l \le g$,

$$\rho_{k,l}B_{i,j}\rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k<i \text{ or } j<k \quad (d_1)\\ \rho_{j,l}^{-1}B_{i,j}^{-1}\rho_{j,l} & \text{if } k=i \quad (d_2)\\ \rho_{j,l}^{-1}B_{k,j}^{-1}\rho_{j,l}B_{k,j}^{-1}B_{i,j}B_{k,j}\rho_{j,l}^{-1}B_{k,j}\rho_{j,l} & \text{if } i<k<j \quad (d_3)\end{cases}$$

(l) *for all* $1 \le i < j \le n,\ 1 \le u \le n,\ u \neq j$ *and* $1 \le t \le b-1$,

$$x_{u,t}B_{i,j}x_{u,t}^{-1} = \begin{cases} B_{i,j} & \text{if } u<i \text{ or } j<u \quad (l_1)\\ x_{j,t}^{-1}B_{i,j}x_{j,t} & \text{if } u=i \quad (l_2)\\ x_{j,t}^{-1}B_{u,j}x_{j,t}B_{u,j}^{-1}B_{i,j}B_{u,j}x_{j,t}^{-1}B_{u,j}^{-1}x_{j,t} & \text{if } i<u<j \quad (l_3)\end{cases}$$

(m) *for all* $1 \le k,u \le n,\ k \neq u,\ 1 \le l \le g$ *and* $1 \le t \le b-1$,

$$x_{u,t}\rho_{k,l}x_{u,t}^{-1} = \begin{cases} \rho_{k,l} & \text{if } k<u \quad (m_1)\\ x_{k,t}^{-1}B_{u,k}x_{k,t}B_{u,j}^{-1}\rho_{k,l}B_{u,k}x_{k,t}^{-1}B_{u,k}^{-1}x_{k,t} & \text{if } u<k \quad (m_2)\end{cases}$$

(n) *for all* $1 \le i < j \le n$ *and* $1 \le s,t \le b-1$,

$$x_{i,t}x_{j,s}x_{i,t}^{-1} = \begin{cases} x_{j,s} & \text{if } t<s \quad (n_1)\\ x_{j,t}^{-1}B_{i,j}x_{j,t}B_{i,j}^{-1}x_{j,t} & \text{if } t=s \quad (n_2)\\ x_{j,t}^{-1}B_{i,j}x_{j,t}B_{i,j}^{-1}x_{j,s}B_{i,j}x_{j,t}^{-1}B_{i,j}^{-1}x_{j,t} & \text{if } s<t \quad (n_3)\end{cases}$$

Proof. The proof works by induction and generalizes those of [GG3] (closed non orientable case) and [B] (orientable case, possibly with boundary components). It uses the following short exact sequence obtained by forgetting the last strand (see [FN]):

$$1 \longrightarrow \pi_1(N_{g,b} \setminus \{z_1,\ldots,z_n\}, z_{n+1}) \xrightarrow{\alpha} P_{n+1}(N_{g,b}) \xrightarrow{\beta} P_n(N_{g,b}) \longrightarrow 1.$$

The presentation is correct for $n=1$: $P_1(N_{g,b}) = \pi_1(N_{g,b})$ is free on the $\rho_{1,l}$'s and $x_{1,t}$'s for $1 \le l \le g$ and $1 \le t \le b-1$. Suppose inductively that $P_n(N_{g,b})$ has the given presentation. Then, observe that $\{B_{i,n+1} \,/\, 1 \le i \le n\} \cup \{\rho_{n+1,l} \,/\, 1 \le l \le g\} \cup \{x_{n+1,t} \,/\, 1 \le t \le b-1\}$ is a free generators set of $\mathrm{Im}(\alpha)$ and $(B_{i,j})_{1\le i<j\le n}$, $(\rho_{k,l})_{\substack{1\le k\le n\\ 1\le l\le g}}$ and $(x_{u,t})_{\substack{1\le u\le n\\ 1\le t\le b-1}}$ are coset representative for the considered generators of $P_n(N_{g,b})$. There are three types of relations for $P_{n+1}(N_{g,b})$. The first one comes from the relations in $\mathrm{Im}(\alpha)$: there are none here, since this group is free. The second type comes from the relations in $P_n(N_{g,b})$: they lift to the same relations in $P_{n+1}(N_{g,b})$. Finally, the third type arrives by studying the action of $P_n(N_{g,b})$ on $\mathrm{Im}(\alpha)$ by conjugation. We leave to the reader to verify that this action corresponds to the given relations. $\square$

Remark 4 What precedes proves that $P_{n+1}(N_{g,b})$ is a semidirect product of the free group

$\pi_1(N_{g,b} \setminus \{z_1, \ldots, z_n\}, z_{n+1})$ by $P_n(N_{g,b})$. Therefore, by recurrence, we get that $P_{n+1}(N_{g,b})$ is an iterated semidirect product of (finitely generated) free groups.

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